Monodromy of Families of Varieties

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Farb Working (Monodromy) Group

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Goals

- Monodromy of families of varieties

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- Examples

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- Examples
- Explicit calculation for a family of cubic surfaces

An elliptic fibration over $\mathbb{P}^1$

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$$\begin{array}{ccc} C_\lambda & \hookrightarrow & E \\ & & \downarrow \\ & & \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{array} \qquad \begin{array}{c} \{(\lambda, p) \mid p \in C_\lambda\} \\ \downarrow \\ \lambda \end{array}$$

C_λ as a ramified cover of $\mathbb{P}^1$

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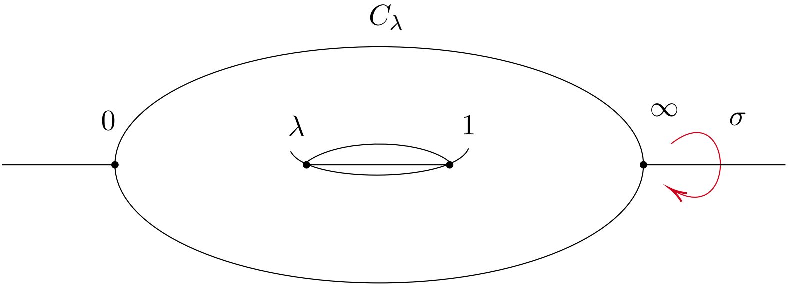
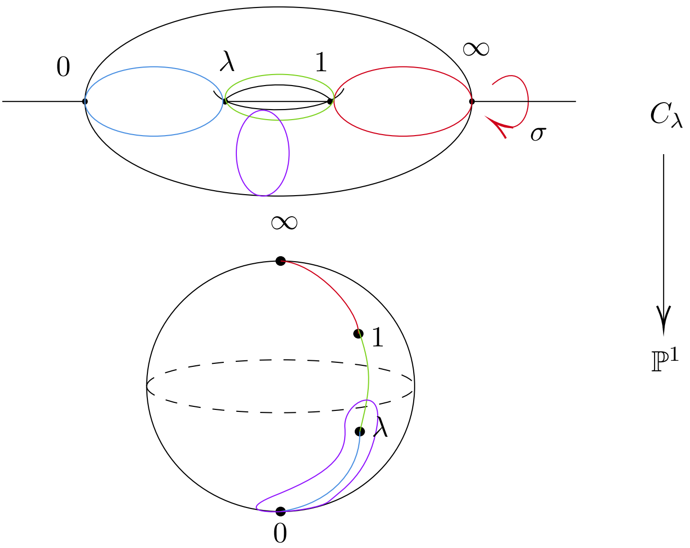
$$\mathrm{Fix}(\sigma) = f^{-1}(\{0,1,\lambda,\infty\})$$

Involution on C_λ



$\mathbb{P}^1$ as a quotient of C_λ



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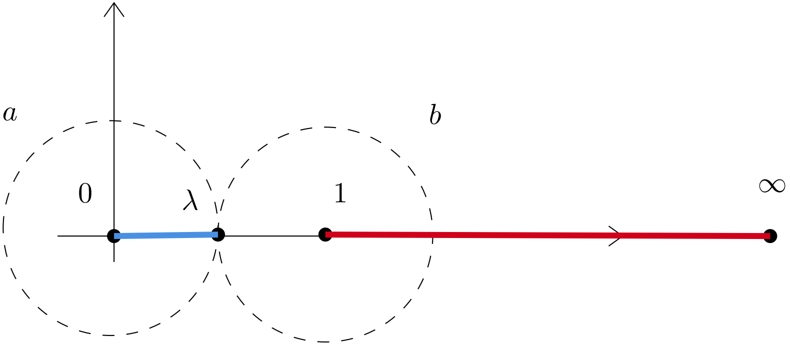
Monodromy

Generators of π_1

$$\pi_1\left(\mathbb{P}^1 \setminus \{0,1,\infty\}, \tfrac{1}{2}\right) \cong \langle a, b \rangle$$



Action of the generators

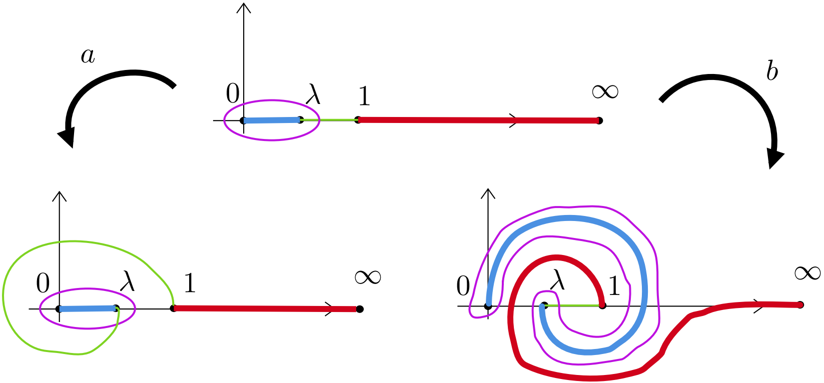


Image of ρ

With respect to our basis of $H_1(C_\lambda; \mathbb{Z})$:

$$\rho(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

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hence

$$\mathrm{Im}(\rho) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \qquad [\mathrm{SL}_2(\mathbb{Z}) : \mathrm{Im}(\rho)] = 12$$

Spaces of smooth projective hypersurfaces

Hypersurfaces in $\mathbb{P}^n$

$$f \in \mathbb{C}[x_0, \ldots, x_n]_d$$

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$$\mathbb{P}^{\binom{n+d}{d}-1} = \{V(f) \subset \mathbb{P}^n \mid f \in \mathbb{C}[x_0, \ldots, x_n]_d\}$$

$$[a_i] \mapsto V\left(\sum_i a_i x^i\right)$$

The complements $\mathcal{U}_{n,d}$

The discriminant variety

$$\{\text{Singular } f\} = \Delta_{n,d} \subset \mathbb{P}^{\binom{n+d}{d}-1}$$

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$$\mathcal{U}_{n,d} = \mathbb{P}^{\binom{n+d}{d}-1} \setminus \Delta_{n,d}$$

What can be said about $\pi_1(\mathcal{U}_{n,d})$?

Fundamental group of $\mathcal{U}_{n,d}$

(Lonne) Presents each $\pi_1(\mathcal{U}_{n,d})$ with $(d-1)^n$ generators.

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(Dolgachev-Libgober)

$$\pi_1\left(\mathcal{U}_{2,3}\right) \cong \mathcal{H}_3\left(\mathbb{Z}/3\mathbb{Z}\right) \rtimes \mathrm{SL}_2\left(\mathbb{Z}\right)$$

where $\mathcal{H}_3\left(\mathbb{Z}/3\mathbb{Z}\right)$ is the Heissenberg group modulo 3.

The universal family $E_{n,d}$

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\begin{array}{ccc}
V(f) \hookrightarrow & E_{n,d} & \qquad \{(f,x) \mid x \in V(f)\} \\
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- If n is even:

$$\mathrm{Im}(\rho) = \begin{cases} \mathrm{Sp}\left(H_{n-1}(V(f);\mathbb{Z})\right) & \text{if } d \text{ is even} \\ \mathrm{SpO}(H_{n-1}(V(f);\mathbb{Z}), q_{V(f)}) & \text{if } d \text{ is odd} \end{cases}$$

Monodromy of $E_{3,3}$

(Klein-Jordan)

$$\mathrm{Im}(\rho : \pi_1(\mathcal{U}_{3,3}) \to \mathrm{Aut}(H_2(V(f);\mathbb{Z}))) \cong W(E_6)$$

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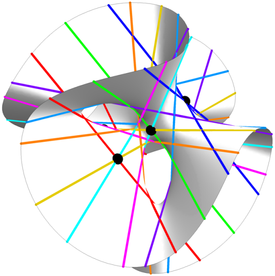

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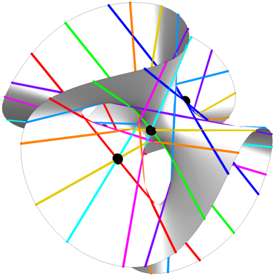
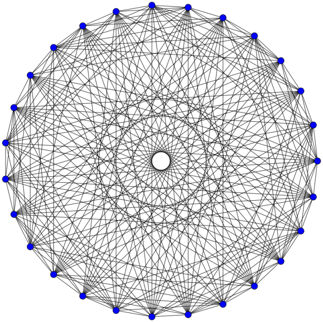


Automorphisms of the 27 lines in a smooth cubic surface.

Monodromy of $E_{3,3}$

Automorphisms of the Schläfli graph.



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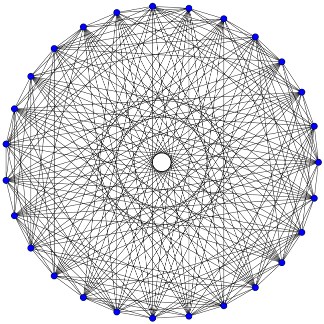

Gives the intersection of the 27 lines in a smooth cubic surface.

Families of branched covers

Ramification over $V(f)$

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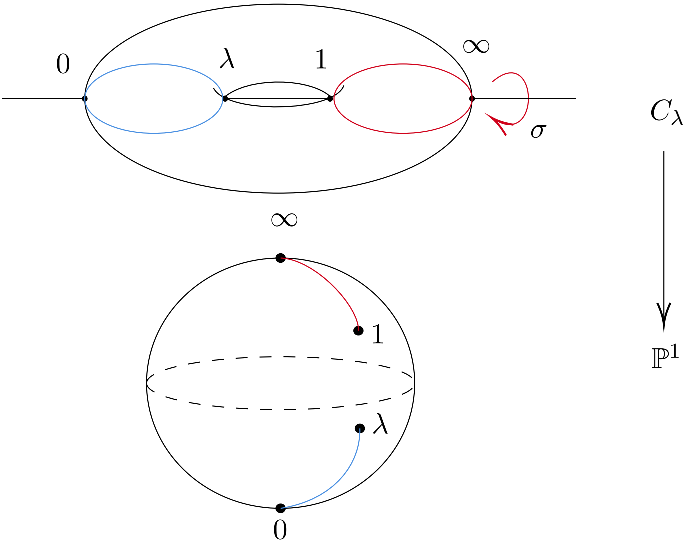
The simplest case: Cyclic covers of $\mathbb{P}^1$

Previous example: Degree 2 cover over 4 points



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$$\rho : \pi_1(\mathrm{Conf}_n(\mathbb{C})) \cong B_n \to \mathrm{Aut}(H^1(X_f;\mathbb{Z})).$$

Branched covers over cubic curves in $\mathbb{P}^2$

Degree $d = 3$ covers

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$$V(f) \subset \mathbb{P}^2 \quad \rightsquigarrow \quad X_f \cong V(w^3 - f) \subset \mathbb{P}^3$$

Monodromy of $\mathcal{E}_{2,3}$

$$\begin{array}{ccc} V(w^3 - f) \hookrightarrow \mathcal{E}_{2,3} & \qquad & \{(f,p) \mid p \in V(w^3 - f)\} \\ \qquad\qquad\quad \downarrow & & \downarrow \\ \qquad\qquad\quad \mathcal{U}_{2,3} & & f \end{array}$$

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$$\rho \text{ preserves } \begin{cases} \langle \cdot, \cdot \rangle_{H^2} & \text{intersection form} \\ K_{X_f} & \text{canonical class} \end{cases}$$

$\mathbb{Z}/3\mathbb{Z}$ Deck group action

$$\mathbb{Z}/3\mathbb{Z} = \langle T \rangle \curvearrowright V(w^3 - f)$$

$$T : [x : y : z : w] \mapsto [x : y : z : \omega^{-1} w] \qquad \omega = e^{\frac{2\pi i}{3}}$$

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$$T^* = \Omega \quad \rightsquigarrow \quad \mathrm{Im}(\rho) \subset C_{W(E_6)}(\Omega)$$

$$|C_{W(E_6)}(\Omega)| = 648$$

Main theorem

(Medrano Martín del Campo)

$$\mathrm{Im}(\rho) \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \cong C_{W(E_6)}(\Omega).$$

A beautiful cubic correspondence:

Lines in X_f and inflection points of $V(f)$

Tangents at the inflection points of f

$$V(f) \cap V(\det \operatorname{Hess}(f)) = I_f \qquad |I_f| = 9$$

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$P \in I_f$

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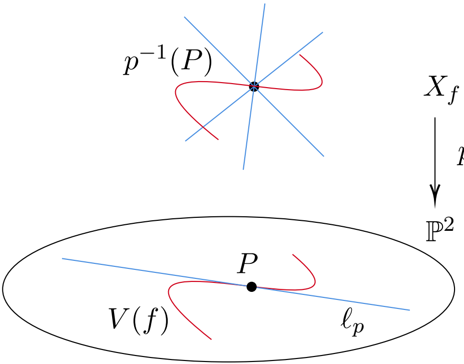
$$P \in I_f \quad \rightsquigarrow \quad \ell_P = V\left(\ker \nabla f_P\right) \subset \mathbb{P}^2$$

3 : 1 line correspondence



$$p^{-1}(\ell_P) = \{3 \text{ lines through } p^{-1}(P)\} \subset X_f$$

3 : 1 line correspondence

$$\{27 \text{ lines in } V(w^3 - f)\} \quad \xleftrightarrow{3:1} \quad \{\ell_P \mid P \in I_f\}$$

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$$\begin{aligned} a \in \pi_1(\mathcal{U}_{2,3}) &\rightsquigarrow \text{Permutation of } P \in I_f \\ &\rightsquigarrow \text{Permutation of } \ell_P \subset \mathbb{P}^2 \\ &\rightsquigarrow \text{Permutation of the 27 lines in } V(w^3 - f) \\ &\rightsquigarrow \rho(a) \end{aligned}$$

Computing $\mathbf{Im}(\rho)$

Two kind of elements in $\pi_1(\mathcal{U}_{2,3})$

(Dolgachev-Libgober) Central extension

$$1 \to \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \to \pi_1(\mathcal{U}_{2,3}) \to \mathrm{SL}_2(\mathbb{Z}) \to 1$$

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$$a \in \pi_1(\mathcal{U}_{2,3}) \begin{cases} \text{Arithmetic} & a \in \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \\ \text{Geometric} & a \in \mathrm{SL}_2(\mathbb{Z}) \end{cases}$$

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$$\rho(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})) \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \quad \text{and} \quad \rho(\mathrm{SL}_2(\mathbb{Z})) \cong \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$$

Generating $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) < \mathrm{Im}(\rho)$

$$f_\lambda = y^2 z - (x-z)(x+z)(x-\lambda z) \qquad \lambda \in \mathbb{C} \setminus \{\pm 1\}$$

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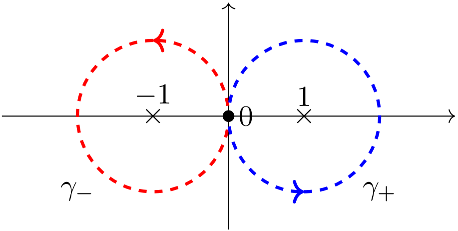

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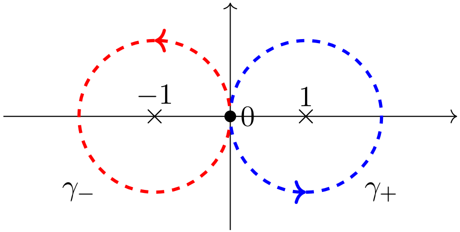

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$$\langle \rho(\gamma_-), \rho(\gamma_+) \rangle = \langle G_1, G_2 \rangle \cong \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$$

Generating $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) < \mathrm{Im}(\rho)$

The Hessian form

$$f_0 \rightsquigarrow f_H = x^3 + y^3 + z^3 - 3\mu xyz$$

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Main theorem

(Medrano Martín del Campo)

$$\mathrm{Im}(\rho) \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \cong C_{W(E_6)}(\Omega).$$